ON THE ENUMERATION OF PATH, UNICYCLIC AND BICYCLIC SEMIGRAPHS WITH ONE OR TWO MIDDLE VERTICES

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Abstract. In this paper we generate non-isomorphic semigraphs by introducing one or two middle vertices in various classes of graphs such as a path, unicyclic graphs with adjacent pendant edges, unicyclic graphs with two non-adjacent pendant edges. Further we proceed to generate non-isomorphic bicyclic semigraphs with one common edge. We obtain the number of non-isomorphic semigraphs using the symmetry property of graphical structures.

1. Introduction

Semigraph is a well known generalization of graphs introduced by E. Sampathkumar [5]. The number of non-isomorphic complete semigraphs on $p$ vertices of the types $K^p_{i}$, $K^p_{i,j}$ and $K^p_{i,j}^{*}$ is given by B. Y. Bam and N. S. Bhave [2]. Further the enumeration of vertex labeled semigraphs containing non-adjacent edges and semigraphs with two $s$-edges is obtained by P. R. Hampiholi and J. P. Kitturkar [10]. This paper deals with the concept of enumeration of unlabeled paths and cycles of semigraphs with exactly one middle vertex. The symmetry property plays a vital role in identifying the non-isomorphic semigraphs.

2. Preliminaries and notations

Definition 2.1. [5] A semigraph $G$ is a pair $G = (V, X)$ where $V$ is a nonempty set whose elements are called vertices of $G$, and $X$ is a set of ordered $n$-tuples, called edges of $G$ denoted by $X = (E_1, E_2, \ldots, E_n)$ of distinct vertices, for various $n \geq 2$, satisfying the following conditions: (i) Any two edges have at most one vertex in common. (ii) Two edges $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ are considered to be equal if and only if (a) $m = n$ and (b) Either $u_i = v_i$ for $1 \leq i \leq n$ or $u_i = v_{n-i+1}$ for $1 \leq i \leq n$.

Thus, the edge $(u_1, u_2, \ldots, u_m)$ is the same as $(u_{m-1}, u_{m-2}, \ldots, u_1)$. The vertices $u_1$ and $u_m$ are the end vertices of $E$, while $u_2, u_3, \ldots, u_{m-1}$ are called the middle vertices of $E$.

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Figure 1.

Example 2.1. The edges of the Figure are \((u_1, u_2, u_3), (u_4, u_5), (u_1, u_6), (u_5, u_6, u_3),\) and \((u_4, u_6)\) is a middle vertex of the edge \((u_5, u_6, u_3)\) whereas it is an end vertex of the edges \((u_1, u_6)\) and \((u_4, u_6)\).

Definition 2.2. A subedge \(E' = (v_{i_1}, v_{i_2}, \ldots, v_{i_k})\) of an edge \(E = (v_1, v_2, \ldots, v_n)\) is a \(k\)-tuple, induced by the set of vertices \(v_{i_1}, v_{i_2}, \ldots, v_{i_k}\) where \(1 \leq i_1 < i_2 < \ldots < i_k \leq n\) or \(1 \leq i_k < i_{k-1} < \ldots < i_1 \leq n\).

Definition 2.3. An \(fs\)-edge in a semigraph \(G\) is an edge or a subedge.

Definition 2.4. A walk in a semigraph \(G\) is an alternating sequence of vertices and \(fs\)-edges \(v_0E_1v_1E_2\cdots v_{n-1}E_nv_n\) beginning and ending with the vertices, such that \(v_{i-1}\) and \(v_i\) are the end vertices of the \(fs\)-edges, \(1 \leq i \leq n\).

Definition 2.5. A \(v_0-v_1\) walk is a trial if any two \(fs\)-edges in it are disjoint.

Definition 2.6. A \(v_0-v_1\) path is a \(v_0-v_1\) trial in which all the vertices are distinct.

Definition 2.7. A pendant edge is an edge containing a pendant vertex.

3. Main Results

Definition 3.1. A semigraph \(1m\)-path is a path on \(n+1\) vertices with exactly one middle vertex and is denoted by \(P(n, 1m)\).

Definition 3.2. A semigraph \(2m\)-path is a path on \(n+2\) vertices with two middle vertices and is denoted by \(P(n, 2m)\).

Definition 3.3. A semigraph is a unicyclic semigraph if it contains a cycle of order \(k\) with a pendant edge and a middle vertex and is denoted by \(C(k, 1p, 1m)\).
Definition 3.4. A bicyclic semigraph denoted by $C(k, l_1p, l_2E, l_3m)$ is a semigraph with two cycles having $l_1$ pendant edges, $l_2$ common edges to the two cycles and $l_3$ middle vertices.

**Case i):** For $l_1 = 0$, $l_2 = 1$ and $l_3 = 1$ the bicyclic semigraph is denoted by $C(k, 1p, 1E, 1m)$.

**Case ii):** For $l_1 = 1$, $l_2 = 1$ and $l_3 = 1$ the bicyclic semigraph is denoted by $C(k, 1p, 1E, 1m)$.

For the sake of clarity we denote the number of non-isomorphic semigraphs as $\mu_s$.

**Theorem 3.1.** The number of non-isomorphic semigraphs $P(n, 1m)$ and $P(n, 2m)$ are given by,

1. $\mu_s \{P(n, 1m)\} = \left\lceil \frac{n}{2} \right\rceil$
2. $\mu_s \{P(n, 2m)\} = \left\lceil \frac{n}{2} \right\rceil + \binom{n-1}{2}$

where $n$ is the number of end vertices and $m$ is the middle vertex.

**Proof.** Consider a path $P_n$ as shown below in Figure 2.

**Case 1:** Path on $n + 1$ vertices that is with $n$ end vertices and exactly one middle vertex without any $(m, e)$ vertices. As path is symmetric about its centre(s) a middle vertex can be inserted in $\left\lceil \frac{n}{2} \right\rceil$ ways. Because choosing an edge $a_1$, $a_2$ is same as choosing an edge $a_{n-1}$, $a_n$ choosing an edge $a_2$, $a_3$ is same as choosing an edge $a_{n-3}$, $a_{n-2}$ and so on. Therefore number of non-isomorphic paths with $n$ end vertices and one middle vertex is as good as selecting an edge up to isomorphism and is equal to $\left\lceil \frac{n}{2} \right\rceil$.

**Case 2:** Path on $n + 2$ vertices with $n$ end vertices, two middle vertices and no $(m, e)$ vertices.

Consider a path on $n$ end vertices without any $(m, e)$ vertices. To generate $P(n, 2m)$ path we need to insert 2 middle vertices in $n - 1$ edges of $P_n$. The number of paths $P(n, 2m)$ is same as the number of ways of inserting 2 middle vertices in $n - 1$ edges of $P_n$. There arise two cases of inserting these two middle vertices. First one is inserting two middle vertices in only one edge. As in case (1), this can be done in $\left\lceil \frac{n}{2} \right\rceil$ ways up to isomorphism. Secondly, inserting 2 middle vertices in two different edges out of
Selection of an edge for the first middle vertex | Corresponding Selection of an edge for the second middle vertex | Number of ways
---|---|---
$a_1$ | $a_2, a_3, \ldots, a_{n-1}$ | $n - 2$
$a_2$ | $a_3, a_4, \ldots, a_{n-1}$ | $n - 3$
$a_3$ | $a_4, \ldots, a_{n-1}$ | $n - 4$
$a_{n-2}$ | $a_{n-1}$ | 1

Table 1.

$n - 1$ edges of $P_n$. This can be done in the following way maintaining the isomorphism as shown in the table below.

Hence the total number of ways in the second case is

\[
1 + 2 + \ldots + n - 2 = \frac{(n-2)(n-1)}{2} = \binom{n-1}{2}
\]

Therefore, the total number of paths

\[
\mu_s\{P(n,2m)\} = \left\lceil \frac{n}{2} \right\rceil + \binom{n-1}{2}.
\]

**Theorem 3.2.** The number of non-isomorphic unicyclic $C(k, 1p, 1m)$ semigraphs with one pendant edge and with exactly one middle vertex is

\[
\mu_s\{C(k, 1p, 1m)\} = 2 + \left\lceil \frac{k-2}{2} \right\rceil,
\]

where $k$ is the order of the cycle.

**Proof.** Consider a unicyclic graph $C_k$ with one pendant edge. Let $v$ be the common vertex to the cycle $C_k$ and the pendant edge. The number of non-isomorphic unicyclic $C(k, 1p, 1m)$ semigraphs is same as the number of ways of inserting one middle vertex in $k$ edges of the cycle $C_k$ along with one pendant edge. One middle vertex can be inserted in the pendant edge in exactly one way. Again one middle vertex can be inserted in two incident pendant edges at $v$ in one way. One middle vertex on the edges of the cycle non incident to $v$ can be inserted in \(\left\lceil \frac{k-2}{2} \right\rceil\) ways taking isomorphism into consideration. Hence the total number of non-isomorphic unicyclic $C(k, 1p, 1m)$ semigraphs is

\[
1 + 1 + \left\lceil \frac{k-2}{2} \right\rceil = 2 + \left\lceil \frac{k-2}{2} \right\rceil.
\]

\[\square\]

**Observation 3.1.** The number of $C(k, lp, 1m)$ is obviously same as $2 + \left\lceil \frac{k-2}{2} \right\rceil$.

**Theorem 3.3.** The number of non-isomorphic unicyclic $C(k, lp, 2m)$ semigraphs with $l$ adjacent pendant edges is

\[
\mu_s\{C(k, lp, 2m)\} = 5 + \left\lceil \frac{k-2}{2} \right\rceil + \frac{(k-4)(k-3)}{2},
\]

where $l \geq 2$.

**Proof.** Consider a unicyclic graph $C_k$ with $l$ adjacent pendant edges. The number of non-isomorphic unicyclic $C(k, lp, 2m)$ semigraphs is same as the number of ways of inserting 2 middle vertices in $k+l$ edges of unicyclic $C(k, lp, 0m)$ semigraph. Following are the ways of inserting 2 middle vertices

1. Both the middle vertices in exactly one way in any one of the $l$ pendant edges. This can be done in ONE way.
(2) Two different pendant edges can be inserted with 2 different middle vertices in exactly ONE way.
(3) One in the pendant edge and the other one in the edge incident to $v$. This can be done in exactly ONE way.
(4) Both the middle vertices can be inserted in only one edge incident to $v$. This can be done in ONE way.
(5) Inserting 2 middle vertices in two different edges of cycle $C_k$ incident to $v$. This can be done in ONE way.
(6) Inserting both middle vertices in only one edge of cycle $C_k$ which is non incident to $v$ in $\left\lceil \frac{k-2}{2} \right\rceil$ ways.
(7) Inserting 2 middle vertices in two different edges of $k-2$ non incident edges of $C_k$. Proceeding as in the case 2 of Theorem 3.1 we get the number of ways of introducing two middle vertices as $\frac{(k-4)(k-3)}{2}$.

Summing up all the assertions (1), (2), (3), (4), (5), (6) and (7) we get the total number of non-isomorphic unicyclic $C_{(k, lp, 2m)}$ semigraphs
\[
= 1 + 1 + 1 + 1 + \left\lceil \frac{k-2}{2} \right\rceil + \frac{(k-4)(k-3)}{2}
\]
\[
= 5 + \left\lceil \frac{k-2}{2} \right\rceil + \frac{(k-4)(k-3)}{2}.
\]

\[\Box\]

**Remark 3.1.** For $l = 1$, except case (2) all the cases remains the same. Therefore, number of non-isomorphic unicyclic semigraphs is
\[
\mu_s \{C_{(k, lp, 2m)}\} = 4 + \left\lceil \frac{k-2}{2} \right\rceil + \frac{(k-4)(k-3)}{2}.
\]

**Theorem 3.4.** The number of non-isomorphic unicyclic $C_{(k, 2p, 1m)}$ semigraphs with two non adjacent pendant edges at distance $d_i$ where $1 \leq i \leq \left\lceil \frac{k}{2} \right\rceil$ is given by
\[
1 + \sum_{i=1}^{\left\lceil \frac{k}{2} \right\rceil} \left\lceil \frac{d_i}{2} \right\rceil + \sum_{i=1}^{\left\lceil \frac{k}{2} \right\rceil} \left\lceil \frac{k-d_i}{2} \right\rceil.
\]

**Proof.** Let the pendant edges be incident at $u_i$ and $v_i$ at distance $d_i$, for $i = 1, 2, \ldots, \left\lceil \frac{k}{2} \right\rceil$. There are three ways of introducing a middle vertex in the edges of unicyclic graph $C_k$.

Introducing the middle vertex in:

**Case (a):** Any one of the pendant edges in exactly ONE way.

**Case (b):** Edges of the cycle $C_k$ which belong to the path $u_i - v_i$ with $d_i \leq \left\lceil \frac{k}{2} \right\rceil$. This can be done in $\left\lceil \frac{d_i}{2} \right\rceil$ ways for $i = 1, 2, 3, \ldots, \left\lceil \frac{k}{2} \right\rceil$.

**Case (c):** Edges of cycle $C_k$ which do not belong to the $u_i - v_i$ path. This can be done in $\sum_{i=1}^{\left\lceil \frac{k}{2} \right\rceil} \left\lceil \frac{k-d_i}{2} \right\rceil$ ways.

Summing up all the cases (a), (b) and (c) we get the total number of non-isomorphic unicyclic $C_{(k, 2p, 1m)}$ semigraphs
\[
= 1 + \sum_{i=1}^{\left\lceil \frac{k}{2} \right\rceil} \left\lceil \frac{d_i}{2} \right\rceil + \sum_{i=1}^{\left\lceil \frac{k}{2} \right\rceil} \left\lceil \frac{k-d_i}{2} \right\rceil.
\]

The following table 3 gives the number of non-isomorphic unicyclic $C_{(k, 2p, 1m)}$ semigraphs at odd and even distances.

\[\Box\]
Theorem 3.5. The number of non-isomorphic bicyclic semigraphs with one edge E in common is \( \mu_s \{ C(k, 1E, 1m) \} = k - 1 \), for \( k \geq 3 \).

Proof. Consider a bicyclic graph \( C_k \). Let us denotes the common edge in \( C_k \) as \( E \). Then there arise three cases in which a middle vertex can be inserted in the bicyclic graph \( C_k \).

Case (a): Inserting a middle vertex in the common edge \( E \) can be done in exactly ONE way.

Case (b): Inserting a middle vertex in the edges adjacent to the common edge \( E \) can be done in exactly ONE way.

Case (c): Inserting a middle vertex in the edges non-adjacent to the common edge \( E \) can be done in \( \lceil \frac{2k - 4}{2} \rceil = k - 3 \).

Summing up all the cases (a), (b) and (c) we get the total number of non-isomorphic bicyclic \( C(k, 1E, 1m) \) semigraphs

\[
= 1 + 1 + (k - 3)
= (k - 3).
\]

□

Conclusion

Generating non-isomorphic semigraphs from graphs is an emerging concept in enumeration of semigraphs. In this paper we have initialized the concept of generating non-isomorphic path semigraphs, unicyclic semigraphs with adjacent and non-adjacent pendant edges and bicyclic semigraphs with one or two middle vertices. Various classes of semigraphs can be further considered for the enumeration of non-isomorphic semigraphs.

References


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